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SOLUTIONS TO THE BASE-AGE VARIANT MODELS

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Abstract. Self-referencing models predict the value of Y at age t as a function of both t and a snapshot observation of $Y = Y_0$ at $t = t_0$, which implicitly integrates the entire environment affecting the development of Y. Common examples of such models are site-dependent height over age models, or site index models, hereafter referred to as site models. These models are often developed using pooled cross-sectional and longitudinal data and describe families of multiple curve shapes.

It is advantageous to formulate these models as algebraic difference equations, which can be referred to as "dynamic equations," with their reference variables describing the environment or site quality. For example, in height modeling, site models predict height as a function of age and a height at a base-age known as the site index.

The base-age specific modeling ideology suggests that curves generated by these models are unique to a particular selection of base-age, at which the input data or site index is defined during the estimation of model parameters. Base-age variant models are designed to capture some of the patterns of curves corresponding to different base-ages through a single formula. The curves generated by this approach vary with base-ages and with various methods in which the models can be applied.

However, the available base-age variant models have been limited in their usage to avoid inconsistent predictions and cannot be considered equations in the algebraic sense since they can show that $1 = 0$. To address this issue, I present a mathematical approach that leads to the derivation of a new type of proper base-age invariant equations, which can be applied in various alternative ways for the same purpose as the base-age variant models, but without creating mathematical inconsistencies.

Keywords: Site models; site index modeling; GADA models; self-referencing functions; base-age invariant; base-age variant; path-invariant.

1 Introduction

Historically, guide curve methods (e.g., Schumacher, 1939) were used to develop simplistic site equations based on temporary plot data. However, these methods are no longer considered viable for contemporary forestry management tools. In more recent years, statistical approaches have been used to develop site models as self-referencing equations (Northway, 1985) with parameters estimated on stem analysis or permanent sample plot data (Biging, 1985). These approaches typically involve fitting multiple curves with a single mathematical equation, where each curve is identified by a curve-specific parameter or another identifier, such as a fitted data point. Parameter estimation methods can be broadly categorized into two groups: traditional base-age specific methods (e.g., Heger, 1973; Curtis et al., 1974) and more recent base-age invariant methods (Bailey and Clutter, 1974; Garcia, 1983).

Base-age specific methods assume that each curve is identified by a specific data point corresponding to an arbitrary base-age, and that the curves are fit with multiple passes through all data points, possibly with an absolute or relative offset. Base-ages are usually selected based on convention and have no direct relevance to the ages subsequently used in model implementation. Conversely, base-age invariant (Bailey and Clutter, 1974) methods assume that the set of curves is fitted to data without restricting any individual curve to pass through any specific point.

Base-age specific parameter estimates depend on the selection of base-ages, whereas base-age invariant methods are unaffected by such selections. Advocates of base-age specific models argue that unique curve shapes corresponding to different base-ages result from a specialization of the model's predictions for use with the given base-age. However, since site models are used with all possible base-ages in practice, the argument for the need of models specialized in any specific base-ages is not rational.

Base-age variant models aim to capture various base-age specific curve shapes for all possible base-ages into a single model form that changes smoothly across all possible base-ages. Examples of such models include those proposed by Goelz and Burk (1992), Huang (1994a, 1994b), Payandeh and Wang (1994, 1995), and Wang and Payandeh (1994, 1995, 1996). However, these models have some serious limitations that are detrimental to many practical applications (Bailey and Cieszewski, 2000). Notwithstanding the claims of their authors, none of these models is base-age invariant due to the flawed method of their development and ill-conditioned mathematical structures.

In this context, I will discuss the specific aspects of the intention behind base-age variant models and propose an improved mathematical conceptualization that will result in a new type of truly base-age invariant models serving the same purpose. By addressing the flawed approach of these previous models, we can create a new approach that will be more accurate and better suited to serve the same intention. The proposed improvement will address the limitations of existing base-age variant models and provide a new approach to base-age invariant models that accurately captures base-age specific curve shapes while remaining mathematically sound.

2 Potential uses of base-age specific and base-age variant models

Assuming that the desired outcome is to reflect unique properties of a specific base-age used in model fitting, practitioners may want to have base-age specific estimates for the parameters. However, even if the resulting model is only applied to stands at the given base-age, practitioners may still want to use the model in alternative ways. For example, they may want to predict heights at harvest age directly from available measurements, or simulate height development in annual or periodic iterations within a larger modeling framework.

Traditional base-age specific models are used in various ways. However, the base-age specific aspect is encoded in the parameter estimates' values and is only relevant to the presumed base-age of the field measurements. The arbitrarily selected base-age of the model remains unchanged in the various model uses, and it does not interfere with the model's use of intermediate calculations or yearly iterations to compute height predictions.

In contrast, existing base-age variant models contain variable base-ages that determine the definition of various curve shapes corresponding to different base-age values. These base-ages are also the reference points for the models and govern the curve shapes through the initial input. Therefore, predicting height at a chosen age, given a measured reference height at a known age, would be appropriate for these models. However, using a predicted height as a new reference height to predict any other height at some other age is inappropriate because it is not based on a measurement but on the model's intermediate calculations. Similarly, using base-age variant models to determine values of the reference height or site index that corresponds to a given curve using the predicted height would also be inappropriate. In contrast, traditional algebraic difference equations and all base-age invariant models solvable for site index are suitable for this purpose.

In short, the base-age variant models are intended to integrate multiple site models with base-age specific parameters into a single functional form, smoothing the differences between models for different base-ages. However, the dependence of curve shapes on base-age applies only to field measurements and not to the model's own predictions. Thus, the base-age dependence is not relevant to the various ways in which the equation might be used in forest management.

The original base-age variant models (as proposed by Goelz and Burk, 1992, and others) were designed to have a general form of:

$$Y \leftarrow f(t, t_0, Y_0) \tag{1}$$

In order to better understand the base-age variant models, I define internal equation base-ages and external base-ages. Internal equation base-ages refer to the initial conditions of implicit algebraic difference equations used in the models, while external base-ages are the statistical bases for the base-age specific parameter values. This distinction is important because it suggests that predictions using the same model and initial conditions, but with different iteration lengths or time scales, will be the same regardless of the base-age used for the statistical parameter values. For example, a prediction for age 150 using measurements at base-age 50 and yearly iterations would be the same as a prediction at age 150 using the same initial measurement and 10-year iterations.

With this distinction in mind, a new type of a base-age invariant equation can be specified as:

$$Y = f(t, t_0, Y_0, \mathcal{Z}) \tag{2}$$

where, Y is the response variable, t is the independent variable, t_0 and Y_0 are the equation initial conditions, and $\mathcal{Z}$ represents the external base-age that governs the different shapes of base-age specific curves. By explicitly including the external base-age in the equation, it becomes clearer that the new models are dependent on the statistical base-age for their parameter values, but that the internal equation base-age is not relevant to the model's predictions for different time scales or iteration lengths; and therefore, the model is a base-age invariant equation with base-age variant parameter estimates.

Overall, understanding the distinction between internal and external base-ages is key to understanding the behavior of base-age invariant equations with base-age variant parameters and how they can be used in practical implementations. By specifying the equation in terms of external base-age, it becomes clearer how the statistical base-age influences the shape of the curves, and how the models can be used flexibly for different prediction scenarios.

In this context, t_0 and Y_0 represent the reference points of the algebraic difference equation, while $\mathcal{Z}$ denotes the base-ages of measurements used in model fitting and its applications. As an illustration, let's consider the Chapman Richards (Richards 1959) model:

$$Y_{(t)} = \alpha \left(1 - e^{-\beta t}\right)^{\gamma} \tag{3}$$

Goelz and Burk (1992) developed a base-age variant model based on eq. (3) that can be defined accurately as a one-way assignment, given by:

$$Y_2 \leftarrow Y_1 \left(\frac{1 - e^{-\alpha\,(Y_1/t_1)^{\beta} t_1^{\gamma} t_2}}{1 - e^{-\alpha\,(Y_1/t_1)^{\beta} t_1^{\gamma} t_1}} \right)^{\delta} \tag{4}$$

where, t_n and Y_n denote an arbitrary age and the height at age t_n, respectively, while α, β, γ and δ are unique parameters for this equation.

The symbol $\leftarrow$ indicates a one-way assignment, meaning that the model (4) is not a mathematical equation in the traditional sense and can lead to inconsistencies if used as such.

This model is designed for direct predictions of height at a specific age, given a height measurement at another age. However, it cannot be used for iterative simulations of height growth, or any other purposes, as the produced predictions will be inconsistent. Therefore, those who subscribe to the base-age variant ideology need to improve the utility of such models to enable their use for other purposes, such as incremental predictions in periodic iterations. In the following section, I demonstrate how to accomplish it.

3 The proposed approach to base-age invariant modeling of the multi-base-age specific responses

Cieszewski and Bailey (2000) present a highly flexible approach for formulating advanced algebraic difference equations, or dynamic equations, that can define models with various types of polymorphism and variable asymptotes. These equations allow for the generation of various curve shapes and are mathematically base-age invariant, meaning that their curves do not change with different selections of the equation's initial conditions. As a result, models derived with this approach will always generate consistent predictions whether used for direct, one-step computations or iterative computations in any yearly or periodic iterations.

It is important to note that the parameter estimates of the dynamic site equation derived with the generalized algebraic difference approach can have base-age specific parameter estimates if fitted inappropriately, as the statistical properties of model parameters depend on the data analysis, not on the equation form. However, they can be used as base-age specific models, as well as dynamic equations. The estimates of parameters are base-age specific if the curves are forced to pass through points determined by the actual data at given base-ages during the fitting process. This can be imposed on any fitting of site models regardless of the equations used. For a single base-age, the base-age specific parameter estimates will have particular values that are characteristic of that base-age. Additionally, the parameter estimates of dynamic equations can be multi-base-age specific (e.g., Clutter et al., 1983, Borders et al., 1988) if the curves are forced through data points at multiple base-ages. In either of these cases, the base-age specific character of the model would be reflected in the parameter estimate values.

To reflect multiple base-ages in the parameter estimates of dynamic equations, modifying functions of the base-ages used in multi-base-age specific fitting can be used to alter the parameters. In such a case, the base-age specific responses of the parameters would be explicit and independent of the equation's initial conditions.

To achieve this, one must explicitly identify the measurements' base-age as it influences the value of model parameters in the explicit site equation during the model conceptualization stage before deriving the final dynamic equation. The external base-ages must then be used to define parameter modifying functions that explicitly define the differences in model parameter estimates for different base-ages. For example, one can assume that the considered base model is eq.(3) and its responses to different base-ages are similar to eq.(4).

To derive equations using the generalized algebraic difference approach, one can define an unobservable variable $\mathcal{X}$ denoting site quality and an explicit measurement base-age $\mathcal{Z}$ used to define the base-age specific changes in the model parameters. Then, using these new explicit variables, an explicit site model defining height yield as a function of site quality $\mathcal{X}$ and age t, and the parameter responses as functions of the measurements' base-age $\mathcal{Z}$ can be formulated. Such an explicit site model is then solved for $\mathcal{X}$ and reformulated to the appropriate implicit dynamic equation using an initial condition solution for $\mathcal{X}$ to replace all its occurrences in the original explicit site equation. In such a scenario, the variable $\mathcal{Z}$ remains explicit in the site model and can be freely set to any arbitrary values. During base-age specific model fitting, it should always be set to the values of the relevant base-ages. On the other hand, during model implementation, this variable should always be set to the values of base-ages of the relevant field measurements. These measurements are those used to force the site curves through their values and should be held constant throughout all computations relevant to this field measurement, changing only for another field measurement restricting another site curve.

For instance, assuming that the base model responses resemble equation (4), one can formulate the following theories related to equation (3):

1. The maximum yield (α) in equation (3) can be theorized as proportional to a power transform of the site quality $\mathcal{X}$, i.e., $\alpha \propto \mathcal{X}^{1/\gamma}$.

2. The slope parameter β in equation (3) that influences polymorphism can be assumed to be proportional to site quality $\mathcal{X}$, i.e., $\beta \propto \phi\mathcal{X}$.

3. The slope parameter β in equation (3) can be proportional to a power transform of the statistically measured base-ages $\mathcal{Z}$, i.e., $\beta \propto \mathcal{Z}^{\delta}$.

By incorporating these relationships into equation (3), the resulting site equation, similar in assumptions to equation (4), takes the following form:

$$Y_{(t,\mathcal{X},\mathcal{Z})} = \mathcal{X}^{1/\gamma}\left(1 - e^{-\zeta\,\mathcal{Z}^{\delta}t + \ln(\phi\mathcal{X})}\right)^{1/\gamma} \qquad (5)$$

where:

- $\mathcal{Z}$ is the explicit statistical measurement of the base-age;

- $\mathcal{X}$ is an unobservable variable that describes theoretical growth intensity or the cross-sectional theoretical explicit variable;

- t is the longitudinal variable such as age or its transformation;

- Y is the response variable of interest such as height, diameter, basal area, or volume; and

- **all parameters** are specific to equation (5) and the expressions derived from this equation.

Next, since $\mathcal{X}$ is an unobservable variable that cannot be measured, it is subsequently replaced by its initial condition solution to the above equation:

$$\begin{aligned}\mathcal{X} &= e^{\zeta\,\mathcal{Z}^{\delta}t}\left(1 \pm \sqrt{1 - 4\,\phi\,e^{-\zeta\,\mathcal{Z}^{\delta}t}Y^{\gamma}}\right)/(2\phi) \qquad (6)\\ &= e^{\zeta\,\mathcal{Z}^{\delta}t_0}\left(1 \pm \sqrt{1 - 4\,\phi\,e^{-\zeta\,\mathcal{Z}^{\delta}t_0}{Y_0}^{\gamma}}\right)/(2\phi)\end{aligned}$$

and the resulting dynamic equation can be expressed as:

$$Y_{(t,t_0,Y_0,\mathcal{Z})} = \Big((1+\mathcal{R})\left(e^{\zeta\,\mathcal{Z}^{\delta}t_0} - e^{\zeta\,\mathcal{Z}^{\delta}(2\,t_0-t)}\right)/(2\phi) + e^{\zeta\,\mathcal{Z}^{\delta}(t_0-t)}{Y_0}^{\gamma}\Big)^{1/\gamma} \qquad (7)$$

In equation (6), the first line gives the general formula for $\mathcal{X}$, and the second line gives the formula when the initial conditions are specified at $t = t_0$ and $Y = Y_0$. Equation (7) gives the base-age invariant dynamic equation, where t is the longitudinal variable (e.g., age), t_0 is the initial value of t, Y_0 is the initial value of the response variable of interest (e.g., height, diameter, basal area, or volume), and $\mathcal{Z}$ can be the explicit statistical measurement's base-age used in the model fitting, which can capture the multi-base-age specific responses of the model, as intended in the original base-age variant models.

As previously stated, $\mathcal{Z}$ is an independent variable used in the regression analysis to describe the base ages of the statistical measurements, and in the model applications to identify different base-age specific responses. Additionally, t_0 represents the age corresponding to Y_0. It is the equation's internal base-age, or initial condition. The following equations hold:

$$\mathcal{R} = \sqrt{1 - 4\,\phi\,e^{-\zeta\,\mathcal{Z}^{\delta}t_0}{Y_0}^{\gamma}}$$

or:

$$Y_{(t,t_0,Y_0,\mathcal{Z})} = \Big(\left(1 + \sqrt{1 - 2{Y_0}^{\gamma}/\mathcal{R}}\right)\left(\mathcal{R} - \mathcal{R}^{(2-t/t_0)}\right) + 2\phi\mathcal{R}^{(1-t/t_0)}Y_0^{\gamma}\Big)^{1/\gamma} \qquad (8)$$

where: $\mathcal{R} = 0.5e^{\zeta\,\mathcal{Z}^{\delta}t_0}/\phi$.

4 Discussion

During base-age dependent regression analysis, the values of $\mathcal{Z}$ and t_0 are always the same because they are assigned different values directly from the data, i.e., measurements. However, in model applications, the values of $\mathcal{Z}$ and t_0 are not always the same. In particular, the values of $\mathcal{Z}$ and t_0 are the same only at the initial measurement input into the model. If the model is iteratively used, the value of $\mathcal{Z}$ remains unchanged until another measurement input, while the values of t_0 and Y_0 can change many times with no issues with the base-age invariant equation. When derived through the generalized algebraic difference approach (Cieszewski and Bailey, 2000), the new base-age invariant models with parameter modifying functions of an external variable, such as fitting data base-ages, are proper well-conditioned algebraic difference equations capable to represent multiple patterns of growth series.

The truly base-age invariant equations derived using the proposed approach, with the general form given in Eq.(2), are expected to be more useful than the original base-age variant models of the type given in Eq.(1). The primary advantages of Eq.(7) or Eq.(8) over models similar to Eq. (4) are outlined below:

First, the proposed base-age invariant equation is mathematically sound from a fundamental point of view and cannot lead to illogical conclusions, such as $1 = 0$, yet it fully satisfies the statistical objectives of the base-age variant models, such as Goelz and Burk (1992).

Second, for any given value of $\underline{\mathcal{Z}}$ held constant, the equation is invariant with respect to t_0 selections and satisfies all the properties of a proper dynamic equation.

Third, the equation can be used in a variety of ways, such as yearly or periodic iterative simulations, which can be useful for implementing it within larger modeling frameworks.

Fourth, the equation is derived through explicit formulation of the theoretical basis of the growth dynamics and statistical objectives.

Fifth, the equation explicitly separates the statistical effects of the measurement base-ages, i.e., the effects of the stochastic predictive variables, from the equation's internal structure as entangled in its implicit definition. This allows for explicit separation of individual base-age specific sub-models relating to different base-ages, as well as opening new opportunities in more intricate and explicit statistical analysis of the effects of stochastic predictive variables.

The new type of base-age invariant equation derived above is a model that can vary curve shapes for different fitting base-ages of the applied measurements and is just a special case of a more general concept of curve-variant models, which unlike the base-age variant models may have some real use for modeling changing growth patterns according to different environmental factors. Many such factors may affect the growth trajectories being modeled, such as elevation, geographical location, or soil conditions (e.g., Kiviste 1997 and 1998), different ecological sites (e.g., Monserud 1984), base height for age definition (e.g., Cieszewski 1994), crowding and self-thinning (e.g., Cieszewski and Bella 1993), and density or mortality (e.g., Tait et al., 1988) and other interactions between different variables.

In summary, the base-age specific parameters of site models are affected by different selections of base-ages, but this dependence is relevant only to the measurements used in the model either during its fitting or during its applications. It is not relevant to any use of intermediate computations within the equation. For example, if a dynamic equation is used with base-age specific regression analysis, the resulting model parameters are base-age specific, but the equation on which the model is based can still be base-age invariant. The resulting model can be specific to any given base-age due to the methodology chosen for the data analysis and not due to an erroneous formulation. Its predictions may always be consistent, whether they are computed directly from one age to another or they are computed in yearly or periodic iterations or in any other way. Moreover, the applied equation should always preserve equality. These are the benefits of the approach presented here.

5 Conclusion

The study's main conclusion is that proper mathematical procedures can be used to derive genuine base-age invariant models with explicitly varying growth trajectories in a similar manner as the base-age variant models are intended to do, but without the pitfalls of the latter. Such models can also be useful for modeling the impact of external environmental variables, such as elevation or climate, on inherent growth and yield patterns. To achieve this, it is recommended to express these models as proper base-age invariant mathematical equations with an explicit external variable (e.g., it could be the base-ages of the data used in multi-base-age specific fitting) modifying the model parameters for its different values.

The ill-conditioned base-age variant models proposed by their pioneers should be avoided as they can hinder the fields of biometrics and growth and yield sciences by proliferating confusion and misrepresentation of model properties. Instead, it is recommended to rely on properly derived and well-behaved mathematical equations that are base-age invariant, and which can achieve the same results as the base-age variant models without any of their negative pitfalls.

For all scientific or operational implementations, it is highly recommended to use in self-referencing modeling formally derived, well-behaved, base-age invariant mathematical equations. To derive such equations one can use any GADA or UTADA (Cieszewski 2021) model (e.g., Cieszewski, 2001, 2002, 2003, Cieszewski et al., 2007) and add to it parameter modifying functions of external variables, including in possibilities those of fitting base-ages.

Disclaimer

With more than 40 years of experience and a keen interest in growth and yield modeling, particularly in self-referencing functions, I believe that the ideology behind the base-age variant models is flawed and not suitable for forest management. The proliferation of these models, as introduced by their originators, is causing confusion about terminology and model functionality, and promoting mathematically incorrect methods that produce ill-conditioned pseudo-equations, which are neither practically useful nor theoretically correct (i.e., can lead to $1 = 0$), in addition to the fact that base-age specific parameter estimates do not capture any useful statistical information (Strub and Cieszewski, 2006), and only the base-age invariant model fitting is truly suitable for self-referencing functions (Cieszewski and Strub, 2018).

To address these issues, I presented a way of developing new type of base-age invariant equations that are mathematically and logically sound, while still achieving the intended purpose of the base-age variant models. By using proper mathematical equations, I believe we can avoid violating mathematical laws, logical rules, and baseless claims of alleged desirable model properties.

In short, my goal is to offer a solution to the problems inherent in the base-age variant model ideology, and to provide an example of a new and improved approach to growth and yield modeling that is based on sound mathematical principles and logical reasoning, and yet can be used for the same purpose of modeling the impacts of multi-base-age specific parameter estimations on predictions of varying biased growth trajectories.

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