

VASIP: VISUALIZING MEASUREMENT AND PROCESS ERRORS IN VASI EQUATIONS BASED ON MONSERUD'S SITE INDEX MODEL

C.J. CIESZEWSKI^{1*}, A.A. WAWRYKOWICZ²

¹*Quantfor Co., Watkinsville, GA, USA*

²*GWN Systems Inc., Edmonton, AB, Canada*

**Corresponding Author*

ABSTRACT. We present VASIP (Variable Base-Age-Site-Index Predictions), software for visualizing error propagation in base-age-invariant, self-referencing site index models, based on Cieszewski's equation (Cieszewski, C.J. 2001. Three methods of deriving advanced dynamic site equations demonstrated on inland Douglas-fir site curves. *Can. J. For. Res.* 31:165–173). VASIP implements the parameters of such published models. The underlying mathematical structure of the VASI model is derived from a fractional base equation representing an improved parameterization of the Monserud-type log-logistic equation. The software displays prediction curves with error-propagation regions that combine (i) the process error approximation derived from the underlying model parameter standard errors, and (ii) the propagation of user-specified input of the measurement error in the reference height at any arbitrary age. The tool enforces model-domain limits encoded in species-specific parameter files (which can be edited for sensitivity tests). We provide the software with examples of parameters for the major commercial tree species in Alberta and Saskatchewan. These parameters and associated statistics are intended for exploratory use. For use with other models based on the same VASI equation, the users can provide their own parameters and fit statistics. We illustrate how the error propagation in the self-referencing equations depends on the distance of the prediction age from the measured height age, the errors in height measurements, and on the process errors reflected in the fitting data growth trajectory inconsistencies, captured by the parameter standard errors. VASIP provides tabular and graphical outputs suitable for management reporting and research and is designed to facilitate technology transfer of base-age-invariant modeling in scientific publishing, operational implementation, and public relations. Given the elusive nature of the self-referencing models, based on implicit equations having no inherent errors at the reference point, this software might be transformative in the understanding of the exceptional prediction power of VASI models that for short-term predictions, by definition, cannot be outperformed by any traditional explicit models.

Keywords: Self-referencing models; Base-age invariance; Measurement error; Process error; Forestry modeling; Growth predictions.

1 INTRODUCTION AND BACKGROUND

Accurate growth-and-yield predictions in forest management are essential for decisions about annual allowable cut and sustainability analysis, investment analysis, and silvicultural planning. In even-aged plantation management and intensive forestry, the most pragmatic proxy for site quality is the *site index*—the dominant or top height at a fixed base age—which enables comparisons between different stands and is essential for long-term

planning and management. The site index is widely used all around the world in operational settings with regional and species characteristics (e.g., base ages 25, 50, or 100 years, depending on growth dynamics) (Beck and Trousdell, 1973; Carmean, 1975).

Traditionally, site index systems were based on families of height–age curves corresponding to a set of fixed base age site index classes. The types of curve systems differ in whether they share a common shape (anamorphic), scaled to different productivity levels, or

allow shapes of curves to vary (polymorphic). Selection of individual curves in the traditional systems was based on selecting the site index corresponding to the growth series. However, modern practice increasingly favors *dynamic, base-age-invariant (BAI)* formulations that allow localization directly from a measured height at *any* age and avoid artifacts tied to a fixed base age (Bailey and Clutter, 1974; Cieszewski, 2001, 2021; Cieszewski and Bailey, 2000). These formulations are nowadays typically derived with the generalized algebraic difference approach (GADA; Cieszewski and Bailey, 2000; Cieszewski and Strub, 2008) and in simpler cases with the algebraic difference approach (ADA; Bailey and Clutter, 1974). In contemporary forest management, GADA is currently the most widely used approach for all dynamic growth and yield models, yielding self-referencing functions consistent with the biological growth processes and convenient for inventory projection without using arbitrary site index classes, but rather using the actual inventory-related measurements directly.

Within this family, VASI (Variable-Age Site Index) models¹—originally developed for Alberta species—provide complex polymorphic, self-referencing site equations with variable asymptotes that have seen wide acceptance and implementation. The equation used in the lodgepole pine model of the first author’s M.F. major paper at the University of British Columbia (Cieszewski, 1988), later published in the Canadian Journal of Forest Research by Cieszewski and Bella (1989), and in the species curves of Cieszewski and Bella (1991) is a canonical example, illustrating how two-coefficient, variable base-age site index formulations can be successfully parameterized for regional applications. These were the first polymorphic VASI models with varying asymptotes matching the flexibility of the Monserud (1984) model.

Earlier successful developments of height-growth models motivated the functional forms used in practice; for example, Monserud (1984) showed that his log-logistic based site model provided a flexible and relevant description of height growth for inland Douglas-fir. The broad relevance of mathematical flexibility of this model in simulating desirable growth patterns motivated the development of the more sophisticated VASI formulation of Cieszewski and Bella (1989, 1991). Cieszewski (2001) shows three ways of deriving advanced VASI models, with desirable well-behaved model properties, using the Monserud (1984) model and the Site Index model pro-

posed there, as an example of the targeted mathematical capability of creating desirable growth patterns. The models of Monserud (1984), Cieszewski and Bella (1989, 1991), and Cieszewski (2001) are all capable of modeling virtually identical growth patterns.

The VASI equation used here was first published at CFS, NoFC, Edmonton, Alberta, for fire-origin stands of lodgepole pine, aspen, and white and black spruce (Cieszewski and Bella, 1991). Its industrial use, in Weldwood of Canada’s timber supply analysis, was described by Cieszewski et al. (1999). Later, others used it for Scots pine in Sweden (Elfving and Kiviste, 1997), tested and recommended it for five pine species in Durango, Mexico (Corral Rivas et al., 2004), and used it for stand height, diameter, and volume models of Estonian state forests (Kiviste and Kiviste, 2009). The dynamic Sitka spruce equation of Cieszewski and Nigh (2002) contains the Cieszewski and Bella (1989) model as a special case. The Alberta black spruce and trembling aspen curves of Cieszewski and Bella (1991) were also compared with curves for northwest Ontario (Carmean et al., 2006).

The program we discuss here was originally developed as a command-line FORTRAN program with hard-coded coefficients for the four Alberta species. Later, its second generation was menu driven but still had hard-coded coefficients and provided height-over-age predictions only. VASIPRED 1.1 for MS-DOS was developed in 1993–1995 by A.A. Wawrykowicz with C.J. Cieszewski at the Northern Forestry Centre, Canadian Forest Service, Edmonton, and used external parameter files. The Windows version offered here was coded in 2019–2020 by A.A. Wawrykowicz, with a C# interface over the original C computations ported to C++. This version is described here as VASIP; the screenshots show its window title, “VasiPred.”

The objectives are to improve evaluation and interpretation of the VASI models, facilitate technology transfer, and make their use accessible to non-specialists. In particular, the program is intended to facilitate evaluation of base-age invariant height predictions in terms of their uncertainty dependent on (i) the quality of data used for model calibration, (ii) model fit to the data, (iii) measurement error associated with the input height, and (iv) the species growth pattern variability. The paper describes the following contributions:

- Prediction curves with combined error propagation envelopes.
- Parameter files with model-domain limits for Alberta and Saskatchewan species.
- Domain checks that protect against extrapolation beyond the site index limits, which the user can relax by editing the ASCII .LIM file.
- Tabular and graphical outputs and saved prediction files.

¹Originally, height curves were plotted for a set of fixed Site Index classes, based on arbitrary selections, and as such they were in a sense Site Index curves, primarily showing how Site Index affects height–age curves. The VASI equations are not restricted to any fixed Site Index concept and therefore we call the curves generated with them just Site Curves, which show how any height–age measurement determines a height–age curve for its corresponding Site.

2 NOTATION, CONCEPTS, CONVENTIONS

We use in this paper the following symbols:

- A – Age (breast-height years);
- S – Site index (m) at the chosen base age;
- A_1, A_2 – The reference and prediction ages;
- H_1, H_2 – Total heights (m) at the reference and prediction ages, respectively;
- H_A – Dominant/top height (m) at age A ;
- $H_{A,S}$ – Height (m) at age A and site index S ;
- θ – Vector of model coefficients (a, b, c);
- e – Height measurement error (m), a hypothetical $\pm$ deviation of the reference height entered by the user;
- a, b, c – Model coefficients;
- s_a, s_b, s_c – Standard errors of a, b , and c ;
- h_b – Base height (1.3 m);
- SI – Site index, the total height (m) at breast-height age 50;
- h_1 – Reference height above the base height, $H_1 - h_b$;
- S_b – Site index above the base height, $SI - h_b$.

The traditional mathematical site index models were based on explicit equations (i.e., $H_{A,S} = f_1(A, S)$), in which the age A and site index S were treated as measurable independent variables. The contemporary variable base-age site index models use implicit equations, such as $H_2 = f_2(A_2, A_1, H_1)$, where only A_2 is treated as an independent variable, while A_1 and H_1 are arbitrary height–age measurements substituting an explicit site variable, treated as unobservable and substituted by a varying model parameter. All explicit models predicting H as a function of A contain, across all A , prediction errors associated with the processes governing H . The self-referencing models are geometrically error-free at the reference point, i.e., they reproduce the supplied reference height analytically exactly (Fig. 1).

At the reference point, in these models, the prediction is set to the value of an observation at an arbitrary age and used to guide the predictions for other ages. This point may contain a measurement error, if it uses an empirical measurement, but the actual process error occurs only increasingly with increasing distance of the prediction from the initial condition value, and its differential depends on how consistent the growth patterns are in the model calibration data. For example, the lodgepole pine model of Cieszewski and Bella (1989) was developed on relatively strong and consistent data in comparison to the trembling aspen model of Cieszewski and Bella (1991). Figure 1 shows the differences in process error propagation between relatively consistent data trends of lodgepole pine and famously inconsistent trends in trembling aspen data. At age 150, with $e = 0$, the band widths are 1.11 m for lodgepole pine and 2.66 m for trembling aspen. The distinction between conditional and unconditional errors is illustrated in Fig. 2.

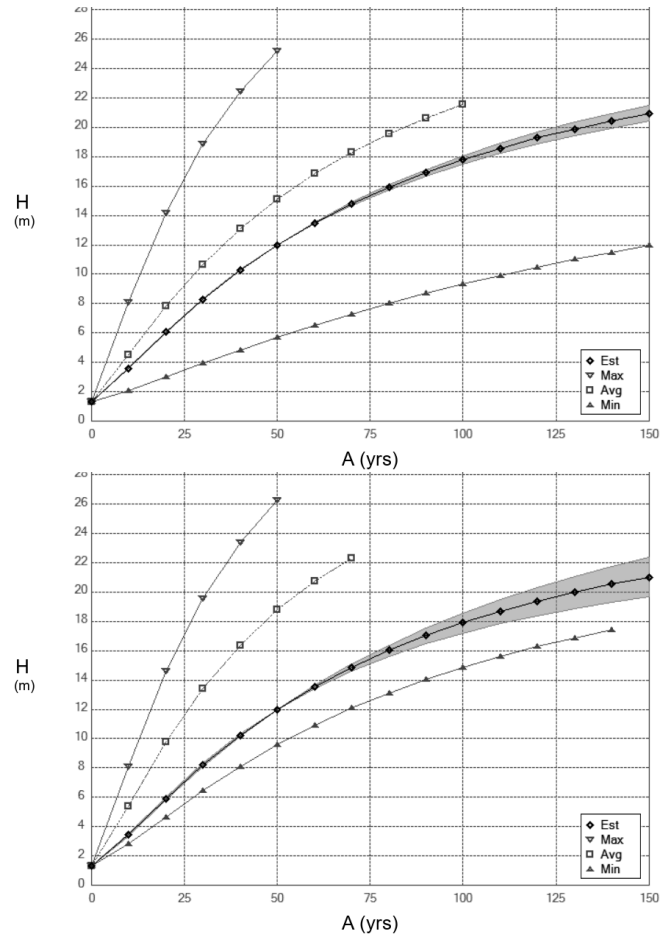


Figure 1: Prediction bands with the height measurement error set to zero for lodgepole pine (top) and trembling aspen (bottom) (reference height 12 m at breast-height age 50, Alberta parameters); the differences in the band widths are due to the process (parameter) errors only.

For this reason, in the operational usage of the self-referencing models, there are no regular error intervals around the self-referencing model predictions, only conditional errors, depending mostly on the distance of prediction time from the time of the input measurement, the measurement error of the initial condition, and the process errors of the model reflected by the departure of the model differential from the differential of the true process. The unconditional regular confidence interval of the self-referencing models is estimated in the parameter estimation phase of the models through a proper fitting procedure, such as the fixed-effect base-age-invariant varying-parameter estimation (Cieszewski et al., 2000; Strub and Cieszewski, 2006). The top images of Fig. 2 (adapted from Cieszewski et al., 2006) show how individual time-series, at different sites (i.e., the unobservable cross-sectional component of the model), comprise the 3D site–age–height surface to be estimated. The

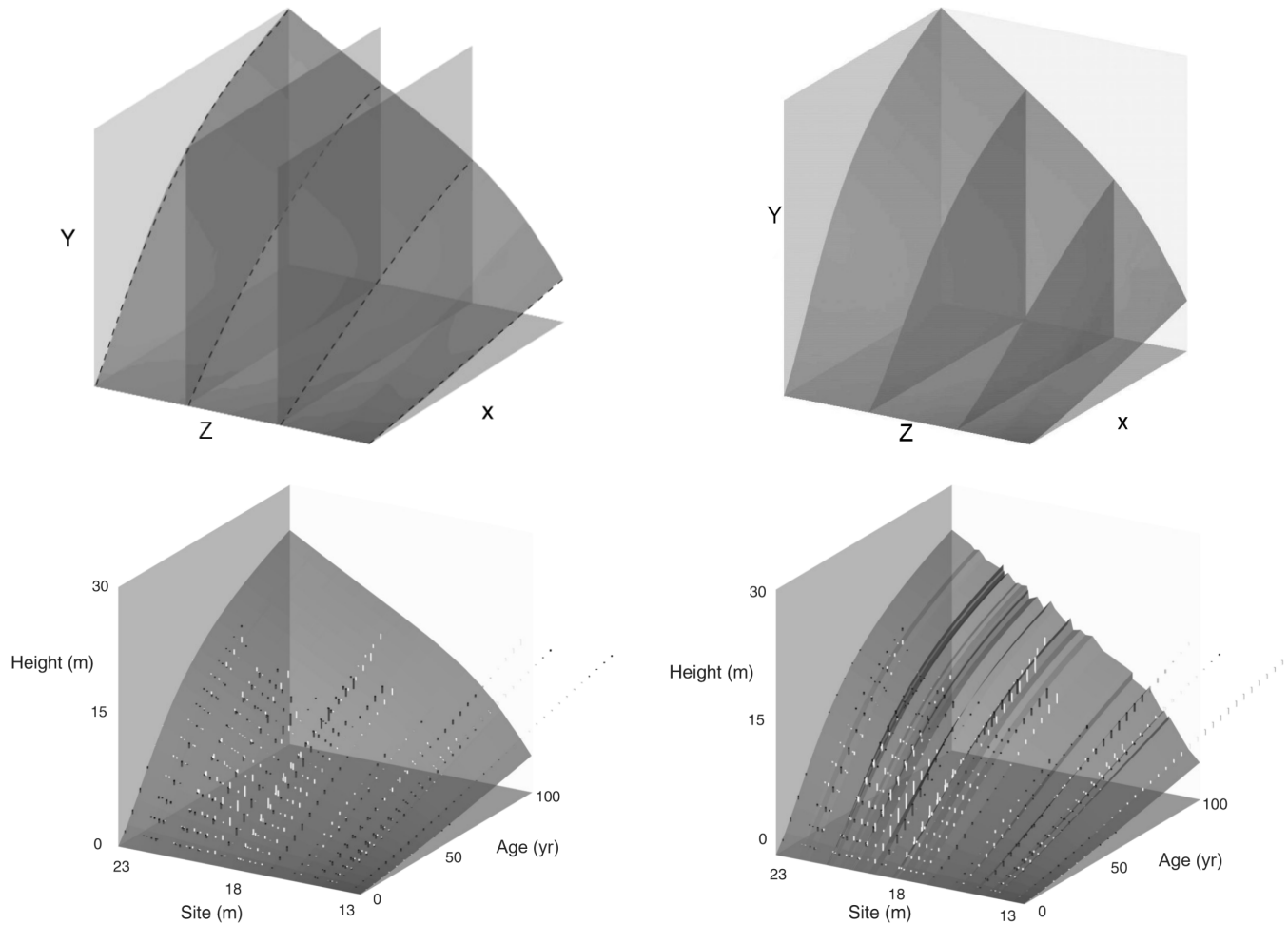


Figure 2: Conditional vs. unconditional errors in site-model predictions. The top images show the site–age–height surface and an individual growth curve selected through a site plane. The bottom images show an unrestricted base-age invariant fit with unconditional residuals (left) and a fixed-base-age fit forced through observations with measurement and environmental errors (right). The top images are adapted from Fig. 2b of Cieszewski et al. (2006); the bottom images are from Fig. 1 of Cieszewski (2003).

bottom images of Fig. 2 (Cieszewski, 2003) show the unconditional errors in fitting the surface with base-age-invariant methods (left) and the erroneous distorted surface in biased fitting of the surface using traditional methods of fixed base-age site index fitting, treating the site as an observable independent variable (right).

In essence, the self-referencing models involve different error considerations during parameter estimation and during the model implementation. In the first case, only minimizing the unconditional errors, including both measurement and process error components, results in parameter estimation giving unbiased model predictions. In the second case, only the conditional errors are considered, with their two main components of measurement error and process error.

Thus we can investigate the errors in predictions of self-referencing models using these two principal sources

of uncertainty. Process errors are reflected in the inconsistencies of the growth trajectories in the fitting data, which are captured by the standard errors of the model parameters; VASIP uses these standard errors as an approximation of the process error and does not add a separate random term for the growth variation of individual stands. Measurement uncertainty arises from errors in the height used to localize the predictions. Height measurement error can be non-trivial (Jurjević et al., 2020; Wang et al., 2019). Both systematic and random height measurement errors have been reported for surveyors using a Vertex hypsometer (Bollandsås et al., 2023). The distance between prediction and reference age also matters, as shown by Fig. 1 and the examples below.

Introduced here, VASIP uses the parameter estimates and their standard errors to show the effect of param-

eter and height measurement errors together. Published models usually do not provide covariance matrices and mostly give parameter estimates and standard errors. The program combines three reference-height values, $H_1 - e$, H_1 , and $H_1 + e$, with three values of each coefficient: its estimate and its estimate plus or minus three standard errors. The lower and upper limits are the minimum and maximum of the resulting predictions, limited to the curves through the minimum and maximum site index of the data. The entered e is a hypothetical $\pm$ deviation, used as entered. Section 4 describes this computation.

3 MODELS

The mathematical formulations discussed here, by Monserud (an explicit equation), and Cieszewski (an implicit equation), differ in numerical properties and appearance, especially as presented in their original sources. Yet, mathematically they are closely related, have historical relevance in continuity of development, and can generate practically identical curves. Historically, Monserud (1984) was an important development in identifying relevant growth patterns for modeling that closely described important empirical data. Conversely, Cieszewski (1988) and Cieszewski and Bella (1989, 1991) were an important step in advancement of mathematical formulations for highly desirable advanced VASI models that could match in performance the complex growth patterns described by the most successful fixed-base-age models, such as the Inland Douglas-Fir site index model.

3.1 Monserud-type log-logistic model

The modified log-logistic site index model of Monserud (1984) was originally derived on the basis of the logistic equation, i.e., $Y = f_1/(1 + f_2 \exp(f_3))$, and in its published form, with implemented site index and log-transform modifications (hence named log-logistic), can be written as:

$$H(A, S) = \frac{b_1 \cdot S^{b_2}}{1 + \exp(\ln b_3 - \ln b_4 \cdot \ln A - \ln b_5 \cdot \ln S)}, \quad (1)$$

where:

- $H(A, S)$ is height at age A and site index S ;
- A is age;
- S is site index at the chosen base age; and
- b_1, b_2, b_3, b_4, b_5 are estimable model coefficients.

We wrote all the exponentiated parameters as logarithmic values for the sake of clarity of further presentation.

Since the logarithms and exponentiations are self-cancelling, the above log-logistic model can be simplified to remove redundancy in mathematical operations. Let E be the argument of the exponential function in the denominator. Then $E = \ln(b_3/(A^{\ln b_4} S^{\ln b_5}))$, and $\exp(E) = b_3/(A^{\ln b_4} S^{\ln b_5})$. Substituting $\exp(E)$ back into $H(A, S)$ and simplifying the outcome we get:

$$H(A, S) = \frac{b_1 A^{\ln b_4} S^{b_2 + \ln b_5}}{A^{\ln b_4} S^{\ln b_5} + b_3}, \quad (2)$$

where:

- b_1, b_2, b_3, b_4, b_5 are estimable species-specific coefficients; and
- H , A , and S are as previously defined.

It is apparent from the above form that the Monserud (1984) model is, as demonstrated in Cieszewski (2001), very similar in principle to the basic formulation of Cieszewski and Bella (1989).

3.2 The Cieszewski (1988) VASI model

This model was developed in the first author's M.F. major paper (Cieszewski, 1988) and was later published for lodgepole pine by Cieszewski and Bella (1989) and applied to the other major Alberta species by Cieszewski and Bella (1991).

The GADA-based advanced VASI family models (the model used here was the first GADA model, although the GADA methodology was formulated later in the first author's Ph.D. thesis, Cieszewski, 1994, and the GADA paper was published in 2000, Cieszewski and Bailey, 2000) provide base-age-invariant self-referencing site-index models with complex polymorphism and variable asymptotes. Generically, the implemented model can be written as:

$$H_2 = f(H_1, A_1, A_2; a, b, c), \quad (3)$$

where:

- H_1 is the measured reference height at age A_1 ;
- H_2 is the predicted height at age A_2 ; and
- a, b, c are the species-specific model coefficients, forming θ .

Exact species statistics (i.e., parameters and their standard errors) and constraints (i.e., data ranges and limitations) are contained in the parameter/limit files provided with VASIP.

The VASI (Variable-Age Site Index) models can predict height directly from a height at any age (any such height may be regarded as SI of its specific age, constituting the model's dynamic base age). In this context, the term height was given two meanings:

- 1 a computed height (H_2) for an age of prediction (A_2), and
- 2 a known height (H_1) at any other age (A_1), used in conjunction with its age (A_1) as a reference point instead of the SI (if $A_1 = 50$ then H_1 is equivalent to SI).

The dynamic site equation implemented in VASIP has the following form (note the similarity to Eq. (2)):

$$H_2 = h_b + \frac{h_1 + d + R}{2 + 4bA_2^{-a}/(h_1 - d + R)}, \quad (4)$$

$$R = \sqrt{(h_1 - d)^2 + 4bh_1A_1^{-a}},$$

where:

- h_1 is the reference height above the base height, $H_1 - h_b$;
- h_b is the base height, 1.3 m in the supplied parameter sets;
- d is bc^{-a} , with $c = 50$ in these sets;
- R is the positive square root defined above;
- a is a positive model coefficient; and
- b , H_2 , A_1 , and A_2 are as previously defined.

The parameters a and b are model coefficients and the derivation constant 50 can be readily replaced by a third coefficient, c . The model coefficients for the Alberta species were determined by nonlinear regressions using stem analysis data (Cieszewski and Bella, 1989, 1991). In the notation of those publications, the program's coefficients are $a = 1 + \alpha$ and $b = 20\beta$.

The connection with the fractional log-logistic form is also seen by fixing the site index base age at 50 and writing height above the base height as:

$$h(A) = \frac{S_b(S_b + d)}{S_b + bA^{-a}}, \quad (5)$$

where:

- $h(A)$ is $H(A) - h_b$;
- S_b is SI $- h_b$; and
- d is $b50^{-a}$.

Dividing numerator and denominator by S_b gives the structure of the simplified Monserud-type form with fixed site exponents and an additive site term d . Equations (4) and (5) are linked through the site index implied by the reference point. With $c = 50$,

$$q = \frac{h_1 - d + R}{2}, \quad H_2 = h_b + \frac{q(q + d)}{q + bA_2^{-a}}, \quad (6)$$

where:

- q is the site index above the base height implied by the reference point (A_1, H_1); and
- the other symbols are as previously defined.

Thus Eq. (5) with $S_b = q$ gives the same predictions as Eq. (4). The program implements the self-referencing equation, Eq. (4), rather than a separate Monserud equation.

3.3 Breast height and the meaning of the heights

The models of Cieszewski and Bella (1989, 1991) were fitted to stem-analysis heights above breast height (1.3 m) against breast-height age, i.e., the number of years since the tree reached breast height. In VASIP breast height is the base height h_b . It is stored for each species in the .PAR file and is 1.3 m in all supplied sets. The program handles it as follows:

- The reference height and the prediction height that the user enters are total heights, measured from the ground. Before Eq. (4) is used, h_b is subtracted from the reference height, $h_1 = H_1 - h_b$, so the equation works with the height above breast height.
- The program adds h_b to every computed height. All heights in the tables, graphs, and saved files are therefore total heights: the predicted heights, the lower and upper limits of the band, the Max, Avg, and Min curves, and the site index.
- The site index, "SI @ 50 yrs.bh.", is the total height at breast-height age 50. The program treats the site index limits in the .LIM files as total heights at this age and checks the site index implied by the input against them (Section 4). A site index given above breast height, such as those in Table 2 of Cieszewski and Bella (1991), is 1.3 m lower than the program's value for the same curve.
- All ages are breast-height ages. At breast-height age 0 the height above breast height is zero, so every curve and both band limits start at 1.3 m. The program evaluates this point at 0.001 years.
- The reference height must be greater than h_b , and the reference and prediction ages must be at least one year. The prediction height used for "bh.Age @ 5.0 m" is also a total height; 5.0 m is 3.7 m above breast height.
- The total age of the tree is not computed. It equals the breast-height age plus the number of years the tree needed to reach breast height. The program gives only a rough estimate of the latter, "Tot.Age @ 1.3 m" (Section 5.5).

For example, in the lodgepole pine example used throughout this paper, the reference height of 12 m at breast-height age 50 is a total height, 10.7 m above breast height, of a tree that reached breast height 50 years earlier. The predicted 17.80 m at breast-height age 100 is also a total height. With the program's estimate of 8.4 years to reach breast height, the total ages are about 58 and 108 years.

3.4 Calibration data and parameter sets

Data used for calibration of the Alberta models were from stem analysis of dominant and codominant trees collected by the Alberta Forest Service (AFS), Weldwood of Canada Limited (Hinton Division), and Forestry Canada (Cieszewski and Bella, 1991). The sample covered each species' commercial range in Alberta. Trees were felled, the total height was measured, and sections for ring (age) counts were obtained at stump, breast height (bh), and at fixed intervals of 1.0–2.5 m above breast height.

Initial data screening for all sample trees was based on the following criteria:

1. non-decreasing ages with increasing heights;
2. a minimum of three acceptable measurements above bh per tree; and
3. a minimum bh age of 50 years at the time of felling.

The final data included 1163 lodgepole pine, 698 white spruce, 282 black spruce, and 276 trembling aspen trees, with breast-height ages from 50 to 260, 250, 190, and 140 years, respectively (Cieszewski and Bella, 1991).

Following the initial successful development of this model, it was implemented to model the major commercial tree species in Alberta (Cieszewski and Bella, 1991) and Saskatchewan (C.J. Cieszewski, unpublished report, 1995). The Alberta set includes lodgepole pine (*Pinus contorta* var. *latifolia* Engelm.), white spruce (*Picea glauca* (Moench) Voss), black spruce (*Picea mariana* (Mill.) B.S.P.), and trembling aspen (*Populus tremuloides* Michx.). The Saskatchewan set contains eleven species and comes from an unpublished 1995 internal report of the first author to the Canada-Saskatchewan Partnership Agreement in Forestry (PAIF), Canadian Forest Service, Prince Albert, Saskatchewan. Table 1 gives the coefficients, standard errors, and data limits of both sets.

The site index limits in Table 1 are the values stored in the program's .LIM files, which the program uses as total heights at breast-height age 50 (Section 3.3). For the Alberta species, these values are the maximum, mean, and minimum site indices of the calibration data, which Cieszewski and Bella (1991, Ta-

ble 2) give above breast height, plus the base height of 1.3 m.

4 ERROR PROPAGATION IN VASIP

4.1 Reference conditioning and the height measurement error

The reference point (A_1, H_1) fixes the local site parameter. It is a condition used by the model, rather than a height to be predicted, and carries no prediction error. For any admissible coefficient values the model returns H_1 when $A_2 = A_1$. The observation used for H_1 may nevertheless contain a measurement error.

The entered height measurement error e is a hypothetical $\pm$ deviation of this reference height. It is used exactly as entered to illustrate how a measurement error changes the predictions in the presence of uncertainty in the global coefficients. It is not a standard deviation and is not multiplied by a statistical factor. A user who wants to examine a deviation of ± 3 m enters 3 m. The reference age is held fixed.

4.2 Process uncertainty

The error-propagation envelopes in VASIP illustrate how uncertainty in the estimated growth relationship and deviations in the reference-height measurement affect conditional predictions. The process (parameter) error component considered here is represented by coefficient perturbations scaled to their estimated standard errors (SEs). These SEs reflect the precision with which the growth relationship has been estimated, incorporating the effects of model-data discrepancies, the amount and distribution of information in the calibration data, and the model's parameterization. Consequently, datasets with comparable underlying growth variability can produce different SEs: more informative data generally establish the trends more precisely and support narrower ranges of coefficient perturbations, whereas limited information supports wider ranges. This dependence on information is appropriate for an envelope illustrating uncertainty in predictions from an estimated relationship. In a regular, identifiable finite-sample least-squares fit using conventional residual-based SEs, exactly zero SEs for all freely estimated coefficients imply zero fitted residual variance; this describes agreement with the calibration data and does not, by itself, establish the accuracy of future predictions. VASIP uses the available estimates and SEs to examine coefficient values at their estimates and at plus or minus three SEs, together with a user-specified deviation in reference height. Each admissible coefficient scenario reproduces its supplied reference height exactly because the local site parameter is recalculated to satisfy that

Table 1: Model coefficients, standard errors, site index values, and data age limits in the Alberta and Saskatchewan parameter sets. The maximum, average, and minimum age limits correspond to the maximum, average, and minimum site index curves, respectively.

Species (code)	<i>a</i>	SE(<i>a</i>)	<i>b</i>	SE(<i>b</i>)	SI (m)		Age limits (years)			
					Max	Avg	Min	Max	Avg	Min
<i>Alberta</i>										
Lodgepole pine (IP)	1.2037	0.0058	1947.49	25.47	25.18	15.13	5.70	50	100	260
White spruce (wS)	1.3235	0.0070	5218.33	81.24	25.75	15.52	6.31	50	105	250
Black spruce (bS)	1.1993	0.0121	2297.46	50.26	17.17	10.29	5.28	50	95	190
Trembling aspen (tA)	1.2645	0.0139	2347.39	64.41	26.23	18.77	9.58	50	70	140
<i>Saskatchewan</i>										
Lodgepole pine (IP)	1.1816	0.0161	1293.85	39.85	19.51	13.97	4.73	100	105	130
Jack pine (jP)	1.1872	0.0155	1358.82	39.98	18.40	13.84	8.72	90	130	60
Black spruce (bS)	1.2193	0.0176	2301.08	85.68	18.24	11.07	5.21	90	140	268
White spruce (wS)	1.3792	0.0170	4915.69	173.62	21.96	14.77	6.20	100	275	80
Manitoba maple (mM)	1.2155	0.0556	2085.26	206.70	18.02	14.02	10.51	50	70	70
Trembling aspen (tA)	1.1857	0.0306	1360.65	77.27	23.21	15.74	7.60	70	110	80
White elm (wE)	1.2112	0.0359	1450.52	107.88	18.27	13.32	8.69	80	120	200
Balsam fir (bF)	1.5219	0.0332	7439.12	461.51	20.12	13.30	6.29	80	90	120
Balsam poplar (bP)	1.1031	0.0263	1350.15	68.51	23.28	16.05	7.01	60	90	194
White birch (wB)	1.2450	0.0282	1697.31	96.30	20.29	13.32	7.61	50	120	245
Tamarack larch (tL)	1.3920	0.0656	1974.46	343.56	16.22	9.74	5.24	50	120	140

Note: $c = 50$ and $SE(c) = 0$ for all species. The base height is 1.3 m. Site index is at breast-height age 50, and ages are breast-height years. Alberta sources: Cieszewski and Bella (1989, 1991). Saskatchewan source: unpublished 1995 report of the first author to the PAIF; for exploratory use. Values of a and $SE(a)$ are rounded to four decimals, and b and $SE(b)$ to two decimals; the program uses the full file values. The black spruce value of a in the file (1.199266) differs in the fifth decimal from $1 + \alpha = 1.1992266$ of Cieszewski and Bella (1991).

condition. Thus, before clipping to the model-domain limits, the parameter contribution to the envelope vanishes at the reference age, while the specified reference-height deviation remains; away from that age, the differences among scenarios reveal how the equation propagates those perturbations. This provides an empirically informed approximation of the magnitude and mechanics of uncertainty propagation without requiring knowledge of the unknown true errors. The resulting widths depend jointly on the estimated coefficient uncertainties, the model's sensitivity, and the reference and prediction conditions. Because marginal coefficient perturbations are combined without their covariance matrix, the envelope does not reproduce the complete joint parameter uncertainty or receive an assigned probability coverage. Its purpose is therefore to demonstrate statistically informed uncertainty propagation, rather than to claim an exhaustive estimate of future growth deviations, consistent with the sensitivity-envelope interpretation in Section 4.3.

The three coefficients a , b , and c are each evaluated at their estimate and at the estimate plus or minus three standard errors. These changes are made independently. They use the estimates and standard errors supplied in the parameter file. The local site parameter is recomputed from the reference point for each combination. In the supplied sets $c = 50$ and $s_c = 0$, so changing c by three standard errors leaves it unchanged.

4.3 The error bands

The program uses all $3^4 = 81$ combinations of $H_1 - e$, H_1 , $H_1 + e$; $a - 3s_a$, a , $a + 3s_a$; $b - 3s_b$, b , $b + 3s_b$; and $c - 3s_c$, c , $c + 3s_c$. There are only 27 distinct curves when $s_c = 0$ and the other changes give distinct values. Each combination is evaluated at breast-height ages 0, 10, 20, ..., 150 years, with 0.001 used in the calculation for age zero. The table lists ages 10–150. The minimum and maximum heights at each grid age form an envelope. This envelope is clamped to the curves through the minimum and maximum site index of the calibration data.

When all combinations pass the domain check, the displayed lower and upper limits are:

$$\begin{aligned} H_L(A) &= \max\left\{\min_{k=1,\dots,81} H_k(A), H_{\min}(A)\right\}, \\ H_U(A) &= \min\left\{\max_{k=1,\dots,81} H_k(A), H_{\max}(A)\right\}, \end{aligned} \quad (7)$$

where:

- $H_L(A)$ is the displayed lower limit at age A ;
- $H_U(A)$ is the displayed upper limit at age A ;
- $H_k(A)$ is the height from combination k ;
- $H_{\min}(A)$ is the curve through the minimum site index in the .LIM file; and
- $H_{\max}(A)$ is the curve through the maximum site index in that file.

The clamping is applied at all grid ages, including ages beyond the age limits used to truncate the displayed data-range curves. The band is a sensitivity envelope of the height and parameter scenarios, not a statistical confidence or prediction interval, although the program labels the table “(95% conf.)”. Entering $e = 0$ gives a parameter-only band.

At the reference age, the coefficients have no effect on each scenario’s reference height. The envelope before clamping is exactly $H_1 \pm e$, with width $2e$, and has zero width when $e = 0$. The examples in this paper have reference-height ranges within the data-range curves, so this equality also holds for their clamped bands. The graph joins the values at the decadal ages rather than computing a continuous band. When the reference age is not a multiple of 10 years, the graph therefore does not show the narrowing of the band at the reference age exactly.

4.4 Domain limits and error messages

The site index implied by the input, $H(50)$, must lie between $SI_{\min} - 0.05$ m and $SI_{\max} + 0.05$ m, where the site index limits are read from `VASIPRED.LIM`. The 0.05 m tolerance allows small deviations from the recorded limits. Because of this tolerance, near the reference age the clamped upper limit can fall below the lower limit; for example, for Alberta lodgepole pine with $H_1 = 25.20$ m at age 50 and $e = 0$, the limits at age 50 are 25.20 and 25.18 m. Prediction table refuses an input outside these limits. For an input above the maximum it reports that the species does not grow that high and gives the maximum height at the reference age. The lower-limit counterpart reports that the tree is too short. The reference and prediction ages must be at least one year, and the reference height must exceed the base height.

The site index check is repeated for each of the 81 combinations. If a combination leaves the limits, the error band cannot be computed. In the Windows version this failure gives no message, and the table limits and the graph band show the minimum and maximum site index curves instead. The data age limits truncate the data-range curves on the graph; they are not checks on the input ages, so predictions beyond the ages of the calibration data are computed without a warning. The graph command does not enforce the input site index refusal, so the input should be checked with Prediction table before using its graph.

The limits can be changed with a standard ASCII editor. File > Load tree species data re-reads the site index limits from `VASIPRED.LIM`. Restart the program if the data age limits have also been changed. Once the original limits are relaxed, the program no longer pro-

tests the user against extrapolation beyond them. There is no override switch and no out-of-domain flag on the output.

Worked example. For Alberta lodgepole pine with $H_1 = 12$ m at $A_1 = 50$ years and $e = 0.10$ m, the site index is 12.00 m. The predicted height at age 100 is 17.80 m, with band limits 17.39 and 18.20 m. At age 150 the height is 20.96 m, with limits 20.28 and 21.64 m. The limits at the reference age are 11.90 and 12.10 m. These values are the table and graph example described below.

5 SOFTWARE DESCRIPTION

5.1 Program and parameter files

The version described here is VasiPred 1.0 (2020). It consists of `VasiPred.exe`, a C# Windows Forms interface, and `VasiModel.dll`, the computational C++ library. It requires 64-bit Windows, the .NET Framework 4.5 or later, and the Microsoft Visual C++ x64 runtime. The program needs no installer. The Alberta set is the active set in the program folder. Both sets are also in the `ParameterSets` folder. Section 5.2 explains how to obtain and install the program.

The binary `.PAR` file contains each species’ coefficients a, b, c , their standard errors, the base height, and a two-character species code. The ASCII `.LIM` file contains a title, species codes and names, maximum, average, and minimum site indices at breast-height age 50, and the three corresponding data age limits. The ASCII `.DEF` file contains a text copy of the coefficients, standard errors, and base heights.

At start-up the program reads fixed filenames `VASIPRED.PAR` and `VASIPRED.LIM` in its working folder. The Console reports this folder as “Executing in:”. To use another parameter set, place its paired files in that folder under these names and restart the program. File > Load tree species data only re-reads `VASIPRED.LIM`; it does not select another set or reload the coefficients. Missing files are not recreated.

If a set is supplied as text, place it in `VASIPRED.DEF` and use Maintenance > Update parameter file (PAR) to create the binary `VASIPRED.PAR`. Then restart the program to load the new coefficients. Maintenance > Create coefficients file (DEF) writes the current parameter file as `VASIPRED.DEF`. New coefficients and their standard errors must describe the same self-referencing equation.

5.2 Availability and installation

VasiPred 1.0 is published with this paper as the supplementary file `VasiPred_1.0.zip` on the article’s page

of the journal web site, <https://mcfns.com/index.php/Journal/article/view/17.4>. The file contains the program, the Alberta and Saskatchewan parameter sets, and a README.txt file with these instructions.

The program needs 64-bit Windows and the .NET Framework 4.5 or later. This framework is part of current Windows versions such as Windows 10 and 11. The Microsoft Visual C++ Redistributable for Visual Studio 2015–2022 (x64) is also required. It is available from Microsoft at https://aka.ms/vs/17/release/vc_redist.x64.exe.

1. Download the ZIP file. Before unzipping, right-click it in Windows. Select Properties, check Unblock, and click OK. This prevents Windows from blocking the program.
2. Unzip it into a folder in which you can write files, e.g., C:\VasiPred. Do not use Program Files. The program writes saved predictions and maintenance files in its own folder.
3. Install the Visual C++ Redistributable if it is not installed yet.
4. Open the extracted VasiPred_1.0 folder and start VasiPred.exe, e.g., by double-clicking it. The Console tab then lists the four Alberta species. If it lists none, the program did not find VASIPRED.PAR and VASIPRED.LIM in its working folder.

To use the Saskatchewan set, copy the three files from ParameterSets\Saskatchewan into the program folder, replacing the Alberta files. Then restart the program. The files in ParameterSets\Alberta restore the Alberta set.

To remove the program, delete its folder. The program writes nothing outside it.

5.3 Main window and menus

The main window has a menu bar, five buttons on the left, and three tabs: Console, Prediction Results, and Prediction Graph. At start-up the Console lists the loaded species with the maximum, average, and minimum site indices (SI1–SI3), their data age limits (Rng1–Rng3), the coefficients with their standard errors in parentheses, and the base height (Hgt). Figure 3 shows the main window and the Main Data Input dialog.

The File menu contains Load tree species data, Show parameters, and Exit. The Predictions menu contains Input data, Prediction table, Prediction graph, Save current prediction, and View prediction files. These five commands are also available through the buttons on the left. The Maintenance menu contains Create coefficients

file (DEF) and Update parameter file (PAR). Figure 4 shows these three menus.

The Help menu contains About. Tooltips explain the buttons and input fields. File > Exit asks for confirmation before quitting. The window's close button also exits the program.

5.4 Input data

Input Data opens the dialog shown in Fig. 3. Tree species selects a model from the loaded set. Reference height is the known total height H_1 that defines the site's curve. Reference age is its breast-height age A_1 . Height measurement error is the hypothetical deviation e around H_1 . Prediction height is the total height for which the user wants a breast-height age, and Prediction age is the breast-height age for which the user wants a height. These are separate requested computations. Section 3.3 explains how the program uses breast height.

The default numerical entries are 12.00 m, 0.10 m, 50.00 years, 5.00 m, and 100.00 years, respectively. The measurement error is limited to 3 m; a larger entry becomes 3.00 m. Accept input stores the values without computing predictions. Cancel restores the previous values. The user then selects Prediction Table or Prediction Graph to compute the results.

5.5 Prediction table

Prediction Table displays the input, computations, coefficients, and decadal heights in the Prediction Results tab. Figure 5 shows the lodgepole pine example with a reference height of 12 m at breast-height age 50 and $e = 0.10$ m.

The line “SI @ 50 yrs.bh.” is the total height at breast-height age 50, here 12.00 m. The line “bh.Age @ 5.0 m” is the breast-height age at which the curve reaches the requested prediction height of 5.0 m, here rounded to 16 years. “Height @ 100 yr” is the height at the requested prediction age, here 17.80 m. “Tot.Age @ 1.3 m” is the approximate total time needed to reach the base height, here 8.4 years. It is calculated by dividing h_b by $H(1) - h_b$, where $H(1)$ is the predicted height at breast-height age 1, i.e., by the predicted growth in the first year above breast height. This is a rough approximation that assumes the same growth rate below breast height as in that year; it has not been validated against data.

The table lists a and b to six decimals and the base height. The coefficient c is not shown in this tab. The decadal table has four columns: BHAge, Height, Min, and Max. BHAge is the breast-height age, Height is the prediction with the entered height and coefficient estimates, and Min and Max are the lower and upper limits described in Section 4. Ages 10–150 are listed.

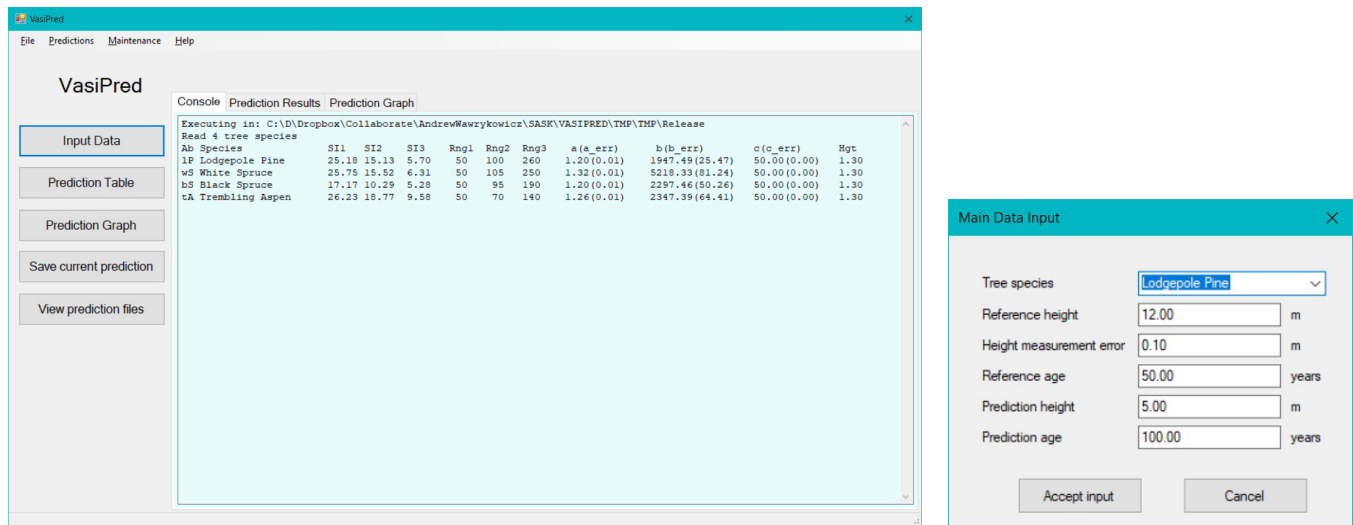


Figure 3: The VASIP main window with the Console tab listing the loaded Alberta parameter set (left) and the Main Data Input dialog (right).

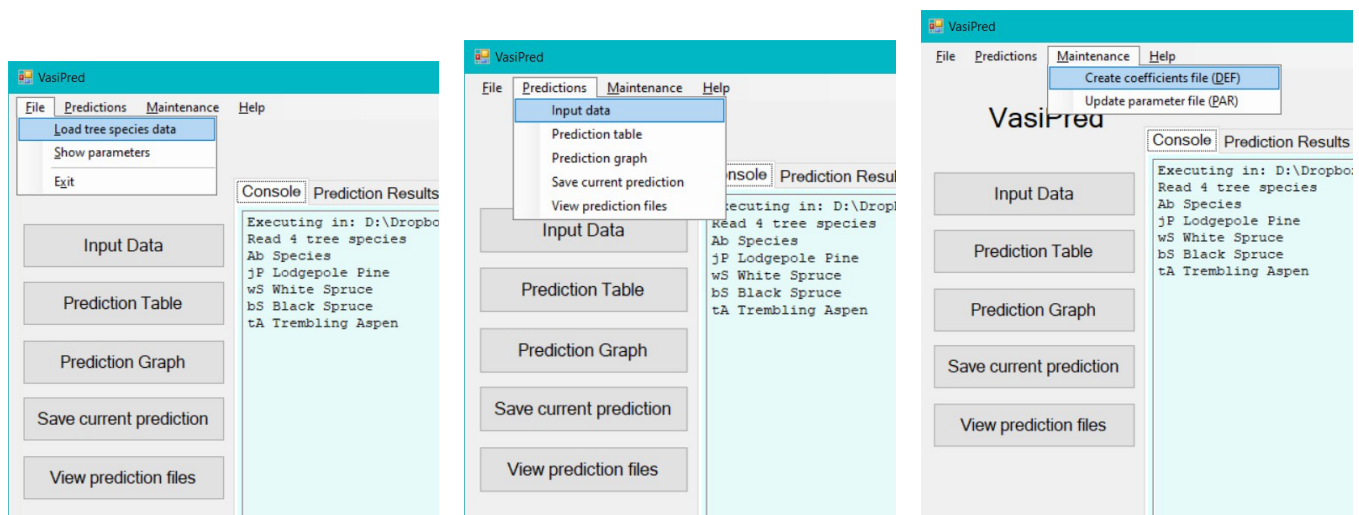


Figure 4: The File (left), Predictions (middle), and Maintenance (right) menus.

The requested age and height computations use the entered values directly rather than interpolation from these rows.

5.6 Prediction graph

Prediction Graph shows the same predictions in graphical form. Figure 6 is the graph for the input of Fig. 5.

The curve labeled Est is the prediction, drawn with white diamonds. The gray region is its error band. The Max, Avg, and Min curves pass through the maximum, average, and minimum site indices of the calibration data at age 50. They have triangle, square, and triangle markers, respectively, with a dashed line for Avg. These data-range curves are drawn up to their corresponding

data age limits. The heights are total heights, so all curves start at 1.3 m at breast-height age 0. The graph background is black and its axes are fixed at 0–150 years and 0–36 m.

5.7 Saving and viewing predictions

After a prediction has been computed, Save current prediction writes an ASCII file in the working folder. The filename is VASIXx00.DAT, where xx is the uppercase species code. An overwrite question is shown if the file already exists. The saved file contains the reference height and age, requested height prediction, site index, entered height measurement error, and decadal Height, Min, and Max values.

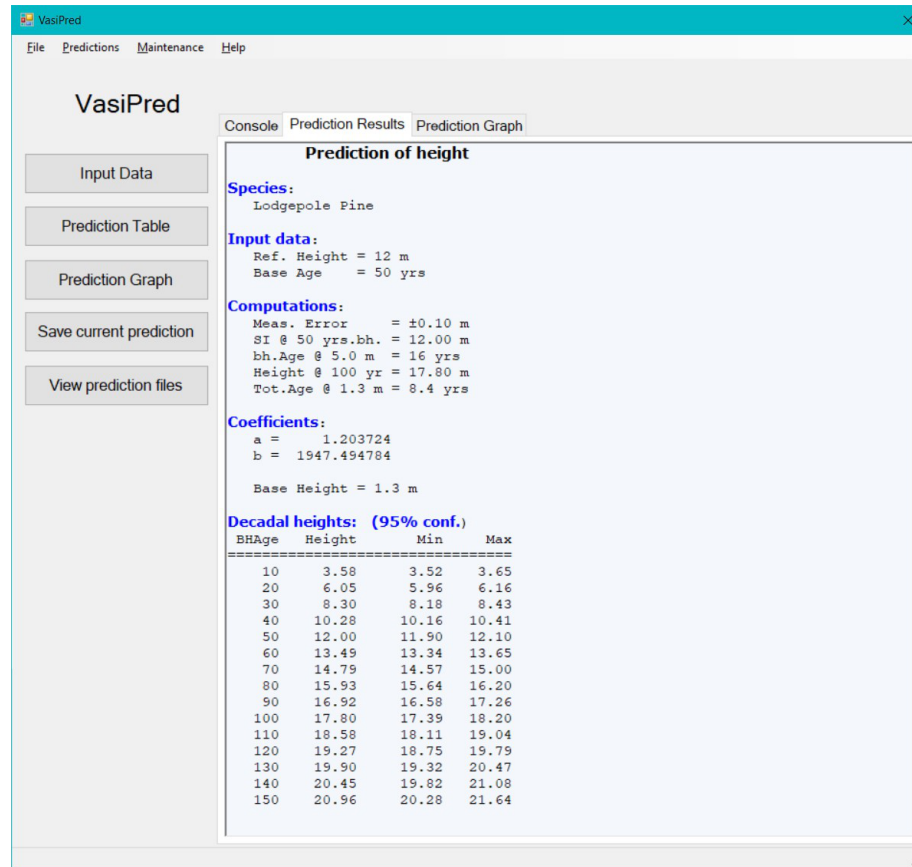


Figure 5: The Prediction Results tab for lodgepole pine with a reference height of 12 m at breast-height age 50 and a height measurement error of ± 0.10 m. The program’s label “(95% conf.)” refers to the sensitivity envelope of Section 4, not to a statistical confidence interval.

View prediction files opens a standard file selection dialog and then a read-only viewer. Saved predictions can be reviewed in this viewer or read as text files. File > Show parameters displays the species last accepted in Input Data, its name and code, the maximum, average, and minimum site indices with their data age limits, a and b with their standard errors and t -ratios (value divided by standard error), and the base height.

5.8 Known limitations

The curves and bands are computed only at decadal ages 0–150, although the user’s requested prediction age and height are computed directly. The graph has fixed axes. The program requires 64-bit Windows and has no built-in printing or export of graphs. In this version a saved prediction is always named VASIXx00.DAT; a different name entered in the Save dialog is not used.

If any of the 81 combinations falls outside the site index limits, the band is not computed and there is no

message. A band that coincides with the curves through the minimum and maximum site index therefore means that the band could not be computed for that input. This can occur with a large entered height error near the limits.

Poorly conditioned models fitted to few, noisy trees can have numerical problems in the age-at-height computation for extreme coefficient combinations. Tamarack larch in Saskatchewan is an example noted in the authors’ earlier software description. The user can choose another prediction height if this computation fails. The Windows version has no numerical error-recovery routines.

6 EXAMPLES

6.1 Effect of the height measurement error

For lodgepole pine from 12 m at age 50, setting $e = 0$ gives the parameter-only band in Fig. 1 (top). With $e = 0.10$ m, the table and graph are shown in Figs. 5

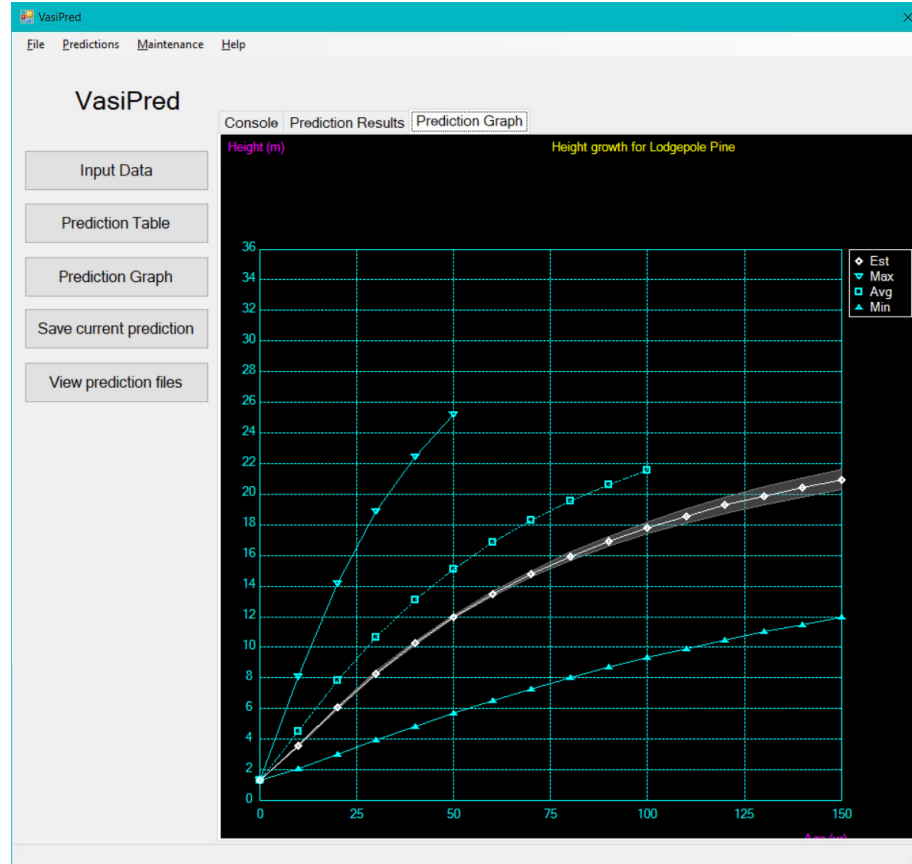
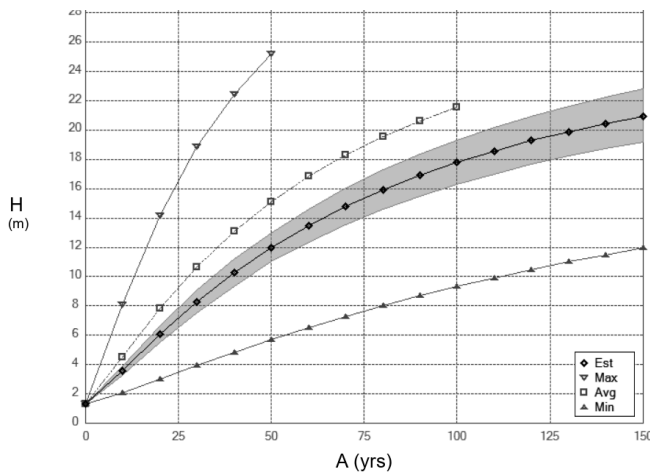


Figure 6: The Prediction Graph tab for the input of Fig. 5.

Figure 7: Lodgepole pine from a reference height of 12 m at breast-height age 50, with $e = 1.0$ m (a hypothetical deviation of ± 1.0 m), Alberta parameters.

and 6. Increasing the hypothetical deviation to ± 1.0 m gives Fig. 7.

Figure 8 includes the intermediate $e = 0.3$ m lodgepole pine case, together with three other Alberta

species. For lodgepole pine the widths at age 150 are 1.11, 1.36, 1.86, and 3.62 m for $e = 0, 0.10, 0.3$, and 1.0 m, respectively. The widths at age 100 are 0.56, 0.81, 1.30, and 3.01 m. At the reference age the band is exactly $2e$ wide in all four cases.

6.2 Differences among species

In Fig. 8 the reference height, age, and measurement error are the same for all four species. Their bands are 0.60 m wide at age 50. At age 150 the widths are 1.86 m for lodgepole pine, 2.61 m for white spruce, 2.96 m for black spruce, and 3.40 m for trembling aspen. Table 1 shows that each of these three other species has larger standard errors of a and b than lodgepole pine. The standard errors depend on the number of sample trees (1163 lodgepole pine, 698 white spruce, 282 black spruce, and 276 trembling aspen trees) and on the consistency of the growth trajectories in the calibration data; the band widths reflect both, as well as the species' curve shapes. The lodgepole pine and trembling aspen difference is al-

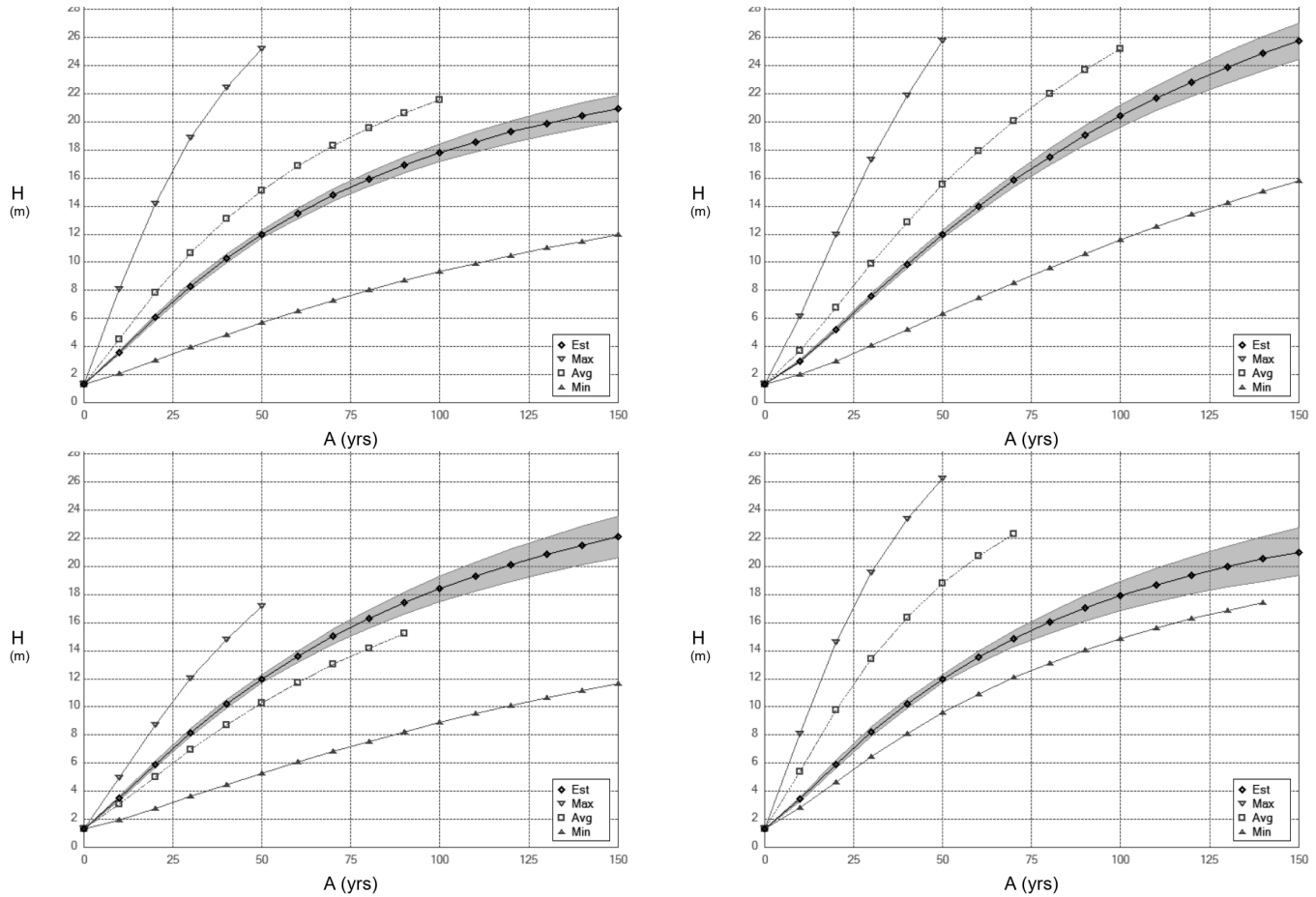


Figure 8: The four Alberta species from a reference height of 12 m at breast-height age 50, with $e = 0.3$ m (a hypothetical deviation of ± 0.3 m): (a) lodgepole pine, top left; (b) white spruce, top right; (c) black spruce, bottom left; and (d) trembling aspen, bottom right.

ready seen when measurement error is set to zero in Fig. 1.

6.3 Reference age

Figure 9 shows lodgepole pine and trembling aspen from 18 m at breast-height age 100, with $e = 1.0$ m. Near the reference age the band is narrowest at that age, where its width is 2.00 m. This location moves with the reference age.

For lodgepole pine in Fig. 9, the band widths at ages 50, 100, and 150 are 2.10, 2.00, and 2.58 m. With the reference at 12 m and age 50 in Fig. 7, the corresponding widths are 2.00, 3.01, and 3.62 m. The two reference points define nearly the same height curve, but the parameter effect is zero at the different reference ages. For trembling aspen in Fig. 9, the widths at ages 50, 100, and 150 are 2.78, 2.00, and 3.31 m. The minimum at the reference age is a local property here, rather than a minimum

over the entire age range: younger heights approach the base height and can have smaller absolute band widths.

7 DISCUSSION

As with any regression models, the models encoded in this software should be applied within the range of the original data. Interpretation of extrapolations should be made with caution and with experience or local knowledge of height–age relationships. The VASIP program facilitates the use of the VASI model for predictions of height growth for different commercial timber species in Alberta and Saskatchewan, and also helps in exploring the influence and importance of different model coefficients. For example, if the user notices that the jack pine curves for Saskatchewan are considerably steeper at high ages than other published curves, the program can be used to investigate whether the two-parameter VASI

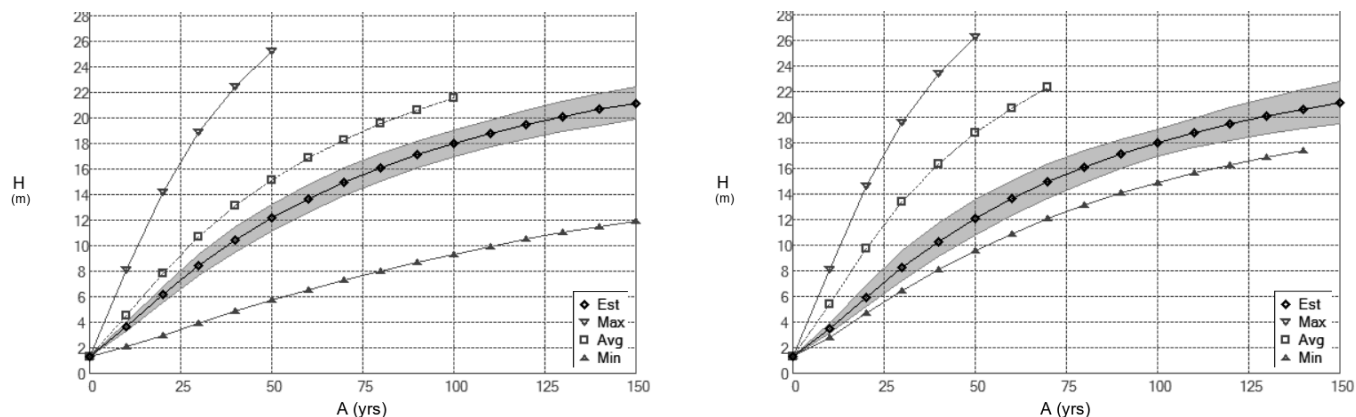


Figure 9: Lodgepole pine (left) and trembling aspen (right) from a reference height of 18 m at breast-height age 100, with $e = 1.0$ m (a hypothetical deviation of ± 1.0 m), Alberta parameters.

model is sufficiently flexible to accommodate a similar shape.

If the user modifies the program's input parameters, the user is also responsible for the output. The standard errors must be updated together with new coefficients. The site index and age limits must also match the data used to develop the new model. Although demonstrated here with height, the same self-referencing form can be used with appropriate coefficients for other variables, such as diameter, basal area, and volume. This is a use of the same mathematical form, not a choice of additional equations in the interface. Upgrades and additional parameter files may be requested from the corresponding author.

Any VASI model, wherever it was developed, reproduces its measured reference height exactly. For a short projection from a recent measurement, the measurement error is therefore the starting uncertainty, and the parameter effect is small near that point. For lodgepole pine with $e = 0$, the band widths at ages 40, 50, and 60 are 0.07, 0.00, and 0.09 m. With $e = 0.10$ m they are 0.25, 0.20, and 0.31 m. These values show the small parameter effect close to the reference age. This property does not remove the need to judge whether the curve is suitable for the stand, but it allows short projections to be localized by the latest measurement. A forester does not have to wait until a stand reaches a fixed base age, such as 50 years, to use the curves. In this sense, traditional explicit models, which treat site index as a measured independent variable (Section 2), do not reproduce a new height measurement in general. When an explicit equation is instead solved for the site index implied by the measurement, it is used as a self-referencing model; for example, Eq. (5) with $S_b = q$ gives the same predictions as Eq. (4).

The displays help users see how the reference measurement and the estimated coefficients affect their pre-

dictions. Their main limitation is that the band is a sensitivity envelope based on published standard errors, rather than a statistical interval. Its width should be interpreted in relation to the chosen height deviation, the parameter set, the reference age, and the calibration data limits.

8 CONCLUSIONS

VASIP provides tabular and graphical predictions from a height measured at any admissible breast-height age. It shows the combined effect of a hypothetical height measurement error compounded with and propagated by the growth dynamics process error, which is reflected in the standard errors of the model coefficients. The Alberta and Saskatchewan parameter sets provide examples for examining these effects among species and reference ages. The program supports forest modeling practice, research, and operational planning by making the behavior of self-referencing predictions easier to examine. The error band should be used as a sensitivity envelope within the model's data range.

SOFTWARE AVAILABILITY

VasiPred 1.0 with the Alberta and Saskatchewan parameter sets is published with this paper as the supplementary file `VasiPred_1.0.zip`, <https://mcfns.com/index.php/Journal/article/view/17.4>. Section 5.2 explains the installation.

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